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Uncertainty Estimation in AoA-based Localization using PCE
Estimation de l'incertitude en localisation par angles d'arrivée avec le développement en chaos polynomial

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Abstract:
In this paper, polynomial chaos expansions (PCEs) are applied to angle-of-arrival-based localization. By applying a polynomial chaos expansion on the least squares estimator, a new positioning method is designed. From this expansion, three methods are discussed to obtain confidence regions of the position. Simulation results show that the proposed methods provide a substantial gain in calculation time compared to a Monte-Carlo simulation, while giving accurate approximations of the exact confidence region.

Résumé:
Dans ce travail, les développements en chaos polynomial sont appliqués à la localisation par angles d'arrivée. Une nouvelle méthode de localisation est proposée, utilisant le développement en chaos polynomial d'un estimateur par moindres carrés de la position. Trois méthodes permettant d'obtenir une zone de confiance de la position à partir de ce développement sont présentées. Les résultats de nos simulations montrent que ces méthodes apportent un gain substantiel en temps de calcul par rapport aux simulations de type Monte-Carlo, tout en donnant des approximations précises de la zone de confiance exacte.

1 Introduction

Accurate localization is one of the fundamental requirements of future Internet-of-Things (IoT) networks. In a cellular infrastructure, high localization accuracy allows network providers to offer additional services related to contextualized information delivery, targeted advertising or security applications. Traditional localization approaches based on the estimation of the signal ToA/TDoA rely on wide communication bandwidths, and can thus not be applied to the localization of IoT nodes (which typically operate with low data rates and low bandwidths). By contrast, angle-of-arrival (AoA) based estimation is less dependent on the bandwidth of the communication system, making it a suitable candidate for IoT localization.
Recently, the deployment of fixed reference nodes, referred to as anchors, which can communicate with the user equipment (UE) to be localized, has been considered [1]. In this work we propose a localization system for a network of densely deployed anchors using AoA measurements at the different anchors. Localization algorithms using AoA measurements have been previously investigated in literature. In [2] the authors use a least squares (LS) estimator, and [3] investigates the use of a linearized LS estimator. While these methods show good efficiency, they do not take into account from the outset the uncertainty of each estimated AoA. We propose to apply polynomial chaos expansion (PCE) theory to the localization of a RF transmitter in order to exploit the AoA measurements at the different anchors, associated with their known uncertainties. This allows us to obtain the location of the transmitter, as well as its statistical distribution, and subsequently, to draw confidence regions. Polynomial chaos expansions allows one to determine the statistical properties of the output of a process, based on the probability density function (PDF) of the input random variables of the process [4]. While the idea to use PCE in AoA-based localisation was initially presented in [5], this contribution presents two original ways of approximating the confidence region using the statistical moments of the position estimate obtained by the PCE. Both these methods are based on the assumption that the position is Gaussian distributed. The first one assumes the two coordinates to be uncorrelated, while the second one introduces the covariance of the two coordinates to take their correlation into account.

2 Angle-of-Arrival based localization

We consider the situation where N anchors of known position $\mathbf{x}_i = (\mathrm{x}_i, \mathrm{y}_i)$ collect the angle-of-arrival measurements θ_i obtained from the signal emitted by one UE of unknown position $\mathbf{x} = (\mathrm{x}, \mathrm{y})$. The measurement of these AoA can be achieved with arrays of antennas by applying the MUSIC algorithm [6]. When more than $N = 2$

anchors are present, this two-dimensional problem is overdetermined. In the case where the AoA measurements are not subject to errors, this deterministic problem can be expressed in the following form, as shown in [2]:

$$\begin{bmatrix} -\mathrm{x}_1 \sin\theta_1 + \mathrm{y}_1 \cos\theta_1 \\ \vdots \\ -\mathrm{x}_N \sin\theta_N + \mathrm{y}_N \cos\theta_N \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 & \cos\theta_1 \\ \vdots & \vdots \\ -\sin\theta_N & \cos\theta_N \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} \tag{1}$$

that can be rewritten as

$$\mathbf{b} = \mathbf{H}\mathbf{x} \tag{2}$$

In a realistic scenario, the measurements will be prone to errors. In that case, the system of equations (1) will not have any solution. To overcome this issue, the least squares estimate of the position $\hat{\mathbf{x}}$ is calculated as:

$$\begin{aligned} \hat{\mathbf{x}} &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{b} && (3) \\ &\equiv \mathbf{H}^\dagger\mathbf{b} && (4) \end{aligned}$$

where † is the pseudo-inverse operator.

3 Angle-of-Arrival based localization using Polynomial Chaos Expansions

Consider that each measurement θ_i is associated with an uncertainty that is also estimated by the anchors. Consequently, the angles-of-arrival can be treated as random variables of known statistical distribution. The least squares estimate of the position of the UE is a function of the random variables θ_i, and is therefore a random variable. We denote this random variable by $\hat{\boldsymbol{x}}$.

3.1 Polynomial Chaos Expansions

The theory of Polynomial Chaos Expansions (PCE) was initially introduced by Norbert Wiener in 1938 with *The Homogeneous Chaos* [7]. The PCE is a surrogate modelling technique, through which a response surface of a computational model is constructed using multidimensional polynomials that are orthogonal with respect to the statistical distributions of the input random variables of this model. As will be explained in the next section, PCE are based on standard random variables, e.g. Gaussian or Uniform, to generate the polynomial basis of the expansion. However, it has been demonstrated that the use of isoprobabilistic transforms makes it possible to apply the theory for any arbitrary random variables [8, 9]. This is usually referred to as *Generalized Polynomial Chaos Expansions.* Recently, the PCE theory has been applied in electromagnetic engineering, in the field of antennas [10, 11, 12], and in propagation theory [13].

3.1.1 Definition of the polynomial basis

Consider a computational model $\mathcal{M}$, with a vector of N input parameters $\boldsymbol{\theta}$, whose output, the response of the model, is $\boldsymbol{x}$:

$$\boldsymbol{x} = \mathcal{M}(\boldsymbol{\theta}) \tag{5}$$

The principle of the PCE is to construct a surrogate model $\hat{\mathcal{M}}$, which is a sum of polynomials on the input variables $\boldsymbol{\theta}$, of the model $\mathcal{M}$, that converges to the model response as the number of polynomials in the sum grows:

$$\begin{aligned} \hat{\boldsymbol{x}} &= \sum_{\alpha\in\mathbb{N}^N} \boldsymbol{c}_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\{\theta_i\}_{i=1}^N) && (6) \\ &= \hat{\mathcal{M}}(\boldsymbol{\theta}) && (7) \end{aligned}$$

In this equation, the coefficients of the expansion are written $\boldsymbol{c}_{\boldsymbol{\alpha}}$. The polynomials $\{\Psi_\alpha(\{\theta_i\}_{i=1}^N)\}_{\alpha\in\mathbb{N}^N}$ form a polynomial chaos basis of the adequate Hilbert space containing $\hat{\boldsymbol{x}}$. These multivariate polynomials are products of univariate polynomials. For each input random variable θ_i, a series of univariate polynomials $\psi_k^{(i)}$, $k \in \mathbb{N}$, are constructed so that they are orthogonal with respect to the scalar product defined by the PDF of θ_i, φ_{θ_i}:

$$\langle \psi_j^{(i)}, \psi_k^{(i)} \rangle = \int \psi_j^{(i)}(u)\psi_k^{(i)}(u)\varphi_{\theta_i}(u)\mathrm{d}u = \gamma_j^{(i)}\delta_{jk} \tag{8}$$

Type of variable	Distribution	Polynomials	Hilbertian Basis $\psi_k(x)$
Uniform	$\mathbf{1}_{]-1,1[}(x)/2$	Legendre $P_k(x)$	$P_k(x)/\sqrt{\frac{1}{2k+1}}$
Gaussian	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$	Hermite $H_{e_k}(x)$	$H_{e_k}(x)/\sqrt{k!}$
Gamma	$x^a e^{-x}\mathbf{1}_{\mathbb{R}^+}(x)$	Laguerre $L_k^a(x)$	$L_k^a(x)/\sqrt{\frac{\Gamma(k+a+1)}{k!}}$
Beta	$\mathbf{1}_{]-1,1[}(x)\frac{(1-x)^a(1+x)^b}{B(a)B(b)}$	Jacobi $J_k^{a,b}(x)$ $\mathfrak{J}^2_{a,b,k} = \frac{2^{a+b+1}}{2k+a+b+1}\frac{\Gamma(k+a+1)\Gamma(k+b+1)}{\Gamma(k+a+b+1)\Gamma(k+1)}$	$J_k^{a,b}(x)/\mathfrak{J}_{a,b,k}$

Table 1 – Standard univariate polynomial families used in PCE [14]

For standard statistical distributions, the associated polynomial families are analytically known. A few examples are given in Table 1. If no standard polynomials are defined for the input distribution, it is possible to define an isoprobabilistic transform to reduce the input variables to components that are distributed according to one of the standard distributions in Table 1 [14].
The multivariate polynomials $\Psi_{\boldsymbol{\alpha}}(\boldsymbol{x})$ are obtained by the tensor product of their univariate counterparts [4]:

$$\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\theta}) \equiv \prod_{i=1}^{N} \psi_{\alpha_i}^{(i)}(\theta_i) \tag{9}$$

3.1.2 Truncation Schemes

The interest of a surrogate model being to reduce the computation time to obtain statistical information on the model response, the expression (6) is not applicable in practice and it needs to be truncated in order to limit the number of coefficients to compute. There are several ways to select the subspace of $\mathbb{N}^N$ to which the multi-index $\boldsymbol{\alpha}$ belongs to. However, we limit ourselves here to the simplest scheme. The standard truncation scheme corresponds to all polynomials in the N input variables of total degree less than or equal to p [14]:

$$\mathcal{A}^{N,p} = \{\boldsymbol{\alpha} \in \mathbb{N}^N : |\boldsymbol{\alpha}| \leq p\} \tag{10}$$

3.1.3 Calculation of the coefficients

Several methods exists to compute the coefficients of the expansion $\boldsymbol{c}_{\boldsymbol{\alpha}}$. The projection method is considered here. This method is a non-intrusive one as it is based on the post-processing of a set of model evaluations, the experimental design. The latter is chosen by an adequate sampling of the input random variables. The orthogonality of the multivariate polynomials, that follows from Eqs. (8) and (9), associated with the definition of the PCE in Eq. (6), directly leads to the following expression for the coefficients:

$$\boldsymbol{c}_\alpha = \frac{\mathbb{E}\left[\hat{\boldsymbol{x}}\Psi_\alpha\right]}{\mathbb{E}\left[\Psi_\alpha^2\right]} \tag{11}$$

In the latter equation, the denominator is known analytically for the standard polynomials in Table 1, whereas the numerator is calculated by an integral:

$$\mathbb{E}\left[\hat{\boldsymbol{x}}\Psi_\alpha\right] = \int_{D_{\boldsymbol{\theta}}} \mathcal{M}(\boldsymbol{\theta})\Psi_\alpha(\boldsymbol{\theta})\varphi_{\boldsymbol{\theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} \tag{12}$$

To numerically evaluate this integral, the usual method is to use the Gauss quadrature method, by which the integral is approximated by a weighted sum:

$$\mathbb{E}\left[\hat{\boldsymbol{x}}\Psi_\alpha\right] \approx \sum_{i=1}^{K} w^{(i)}\mathcal{M}(\boldsymbol{\theta}^{(i)})\Psi_\alpha(\boldsymbol{\theta}^{(i)}) \tag{13}$$

The set of weights $w^{(i)}$, and quadrature points $\boldsymbol{\theta}^{(i)}$ (the *experimental design*) correspond to the roots of the used polynomial (see [15] for details).

3.1.4 Post-Processing

The interest of the Polynomial Chaos Expansion theory is to obtain statistical informations on the model response with less computational effort than by a Monte-Carlo simulation of the actual model. It can easily be demonstrated that the mean and the variance of the model response are respectively given by:

$$\mu_{\hat{\boldsymbol{x}}} = \mathbb{E}[\hat{\boldsymbol{x}}] = \boldsymbol{c}_0 \tag{14}$$

$$\sigma_{\hat{\boldsymbol{x}}}^2 = \mathrm{Var}\left[\sum_{\alpha\in\mathcal{A}\setminus\mathbf{0}} \boldsymbol{c}_\alpha \Psi_\alpha\right] = \sum_{\alpha\in\mathcal{A}\setminus\mathbf{0}} \boldsymbol{c}_\alpha^2 \|\Psi_\alpha\|^2 \tag{15}$$

However, in some applications, it might be necessary to have more statistical informations on the model response. The PCE of the model response may be used to derive an approximation of the probability density function (PDF) of the model response.

3.2 Uncertainty on the position using PCE

3.2.1 First method: Monte-Carlo on the PCE

To obtain an approximation of the PDF of the position of the UE to locate, using the PCE (6), the simplest way is to perform a Monte-Carlo simulation on the PCE itself. For this purpose, a large set of input vectors $\{\boldsymbol{\theta}_k, \quad k = 1, ..., N_{MC}\}$ is generated according to the statistical distribution of the input variables $\varphi_{\boldsymbol{\theta}}$. Then, the position corresponding to each of these AoA samples is obtained via the least squares estimator. The resulting large set of positions allows one to draw the two-dimensional statistical distribution of the position. From this PDF, the confidence region is simply obtained by selecting all points corresponding to a value of the PDF greater than a certain threshold that leads to the desired probability. This method gives very accurate results but requires a considerable computational effort, since a Monte-Carlo calculation is used. The gain in computational time of this first method compared to a simple Monte-Carlo calculation on the least squares position estimator is not significant. However, for more complex estimators, the use of PCE can significantly reduce the computational time needed to obtain similar results.

3.2.2 Second method: Ellipse

In order to exploit the potential of the PCE in terms of gain in computational time, one can make use of the mean and variances of the position estimate obtained with Eqs. (14) and (15). These values are the first and second order statistical moments of the distribution of the position coordinates. By assuming that the position coordinates $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are uncorrelated, and that both distributions are Gaussian, one can obtain an elliptic approximation of the confidence region. As the coordinates are assumed uncorrelated, the principal axes of the resulting ellipse are parallel to the x-axis and y-axis, respectively. The computational cost of this method is greatly reduced compared to the previous one, since no Monte-Carlo calculation is required. Indeed, once the PCE is calculated, the mean and variance of each coordinate is immediately obtained.

3.2.3 Third method: Oriented Ellipse

As seen previously, the second method considers that the position coordinates are uncorrelated and results in an ellipse whose principal axes are parallel to the x and y-axes. Therefore, the quality of this approximation depends on the choice of the axes in the definition of the problem. To cope with this issue, we propose to calculate the covariance between $\hat{x}$ and $\hat{y}$ and to use it for drawing the oriented ellipse. This oriented ellipse is obtained through the diagonalisation of the covariance matrix, and is therefore independent of the reference frame. By 6, the estimation of the x- and y-coordinates are expanded on the same polynomial chaos basis, but separately:

$$\hat{x} = \sum_{\alpha\in\mathcal{A}} c_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\{\theta_i\}_{i=1}^N) \tag{16}$$

$$\hat{y} = \sum_{\alpha\in\mathcal{A}} d_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\{\theta_i\}_{i=1}^N) \tag{17}$$

In those equations, $c_{\boldsymbol{\alpha}}$ and $d_{\boldsymbol{\alpha}}$ are the coefficients of the expansion of the estimates of the x- and y-coordinates, respectively. The covariance $R_{\hat{x}\hat{y}}$ is then derived using the orthogonality of the polynomials of the basis, leading to an expression similar to Eq. (15):

$$
\begin{aligned}
R_{\hat{x}\hat{y}} &= \mathrm{Cov}(\hat{x}, \hat{y}) && (18)\\
&= \mathbb{E}\left[(\hat{x} - \mu_{\hat{x}})\,(\hat{y} - \mu_{\hat{y}})\right] && (19)\\
&= \mathbb{E}\left[\left(\sum_{\boldsymbol{\alpha}\in\mathcal{A}\setminus\mathbf{0}} c_{\boldsymbol{\alpha}}\Psi_{\boldsymbol{\alpha}}(\{\theta_i\}_{i=1}^N)\right)\left(\sum_{\boldsymbol{\beta}\in\mathcal{A}\setminus\mathbf{0}} d_{\boldsymbol{\beta}}\Psi_{\boldsymbol{\beta}}(\{\theta_i\}_{i=1}^N)\right)\right] && (20)\\
&= \mathbb{E}\left[\sum_{\boldsymbol{\alpha}\in\mathcal{A}\setminus\mathbf{0}}\sum_{\boldsymbol{\beta}\in\mathcal{A}\setminus\mathbf{0}} c_{\boldsymbol{\alpha}}\Psi_{\boldsymbol{\alpha}}(\{\theta_i\}_{i=1}^N)d_{\boldsymbol{\beta}}\Psi_{\boldsymbol{\beta}}(\{\theta_i\}_{i=1}^N)\right] && (21)\\
&= \sum_{\boldsymbol{\alpha}\in\mathcal{A}\setminus\mathbf{0}}\sum_{\boldsymbol{\beta}\in\mathcal{A}\setminus\mathbf{0}} c_{\boldsymbol{\alpha}}d_{\boldsymbol{\beta}}\,\mathbb{E}\left[\Psi_{\boldsymbol{\alpha}}(\{\theta_i\}_{i=1}^N)\Psi_{\boldsymbol{\beta}}(\{\theta_i\}_{i=1}^N)\right] && (22)\\
&= \sum_{\boldsymbol{\alpha}\in\mathcal{A}\setminus\mathbf{0}} c_{\boldsymbol{\alpha}}d_{\boldsymbol{\alpha}}\,\mathbb{E}\left[\Psi_{\boldsymbol{\alpha}}^2(\{\theta_i\}_{i=1}^N)\right] && (23)
\end{aligned}
$$

Then, the covariance matrix Σ is constructed from Eqs. (15) and (23):

$$
\Sigma = \begin{bmatrix} \sigma_{\hat{x}}^2 & R_{\hat{x}\hat{y}} \\ R_{\hat{x}\hat{y}} & \sigma_{\hat{y}}^2 \end{bmatrix} \tag{24}
$$

The eigenvalues and eigenvectors of this matrix, obtained simply by diagonalising it, allow one to draw the oriented ellipse. The principal axes of this ellipse are defined by the eigenvectors of the covariance matrix, while the length of the minor and major axes are related to the eigenvalues of the covariance matrix.

4 Results

In the calculations, $N = 3$ anchors have been considered, each of them making an independent AoA estimation. Firstly, Gaussian distributed estimation errors were considered. To the best of our knowledge, there is no precise model for uncertainty distribution in AoA estimation in the literature. It is a common assumption to use Gaussian distribution for uncertainties. This assumption is not restrictive since by using isoprobabilistic transforms, the PCE can be applied to any arbitrary input distribution. The expansion of the position coordinates as a function of the angles has been calculated up to the order 4, with a standard truncation scheme. The choice of the order 4 was made simply by observing the confidence regions obtained for orders up to 5. As there was no significant difference between the CR obtained with the order 4 and 5, the order 4 was selected as a compromise between computational time and precision. In the Monte-Carlo calculation, 20000 runs of the least squares position estimator were used, while 125 were needed to obtain each of the two coordinates PCE. In the first method, we used 20000 realizations of the PCE to draw the confidence region. In comparison, the methods 2 and 3 are more efficient since they only require to compute the mean and the variance of both coordinates - as well as the covariance in method 3 - from Eqs (14), (15) and (23). In the first simulation, the AoA at each anchor was defined by a Gaussian random variable, centred on the actual AoA, and of standard deviation equal to 5°. The results of the three methods are given in Fig. 1, and compared to a Monte-Carlo calculation. We observe that the confidence region presents a stretched shape in the direction of the anchors. An isotropic distribution of the anchors around the UE would lead to a more circular confidence region.

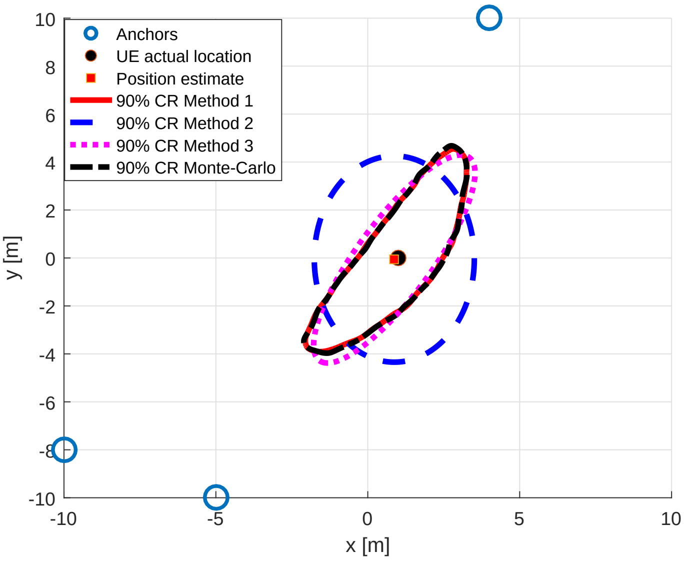

Figure 1 – 90% confidence regions, obtained by the three methods, and assessed by a Monte-Carlo calculation. The three AoA's are defined with a 5° standard deviation.

Then, the standard deviation of the AoA distribution on the anchor located at $(4, 10)$ has been changed to observe its effect on the confidence region. In Fig. 2, this standard deviation was set to 1°, while in Fig. 3 it was set to 15°. In both cases, the other parameters, including the standard deviations of the two other anchors, were identical to the first simulation. We observe in Figs. 2 and 3 that the shape, the size and the orientation of the confidence region is affected by the standard deviation. The oriented ellipse method allows us to almost perfectly cope with these modifications, in contrary to the classical, unoriented, method.

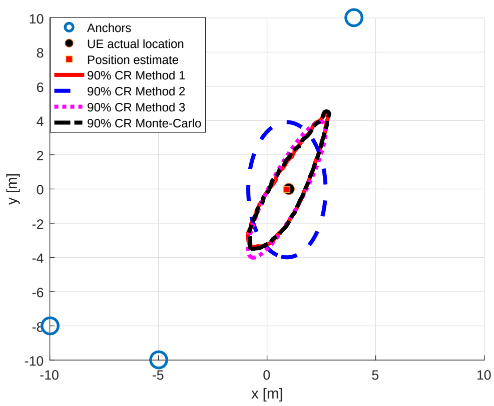

Figure 2 – 90% confidence regions, obtained by the three methods, and assessed by a Monte-Carlo calculation. The standard deviations are 5, 5 and 1°, respectively.

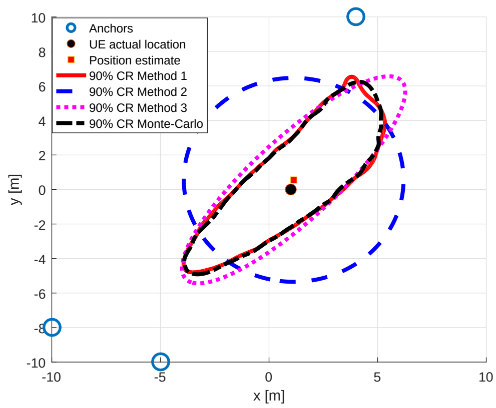

Figure 3 – 90% confidence regions, obtained by the three methods, and assessed by a Monte-Carlo calculation. The standard deviations are 5, 5 and 15°, respectively.

5 Conclusion

A new two dimension position estimation method based on polynomial chaos, least squares estimator, and AoA measurements has been proposed. Compared to traditional positioning methods, it presents the advantage of taking into account the uncertainties on the AoA estimations from the outset. This allows one to compute the covariance matrix of the position estimation, and consequently, confidence regions.

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